Universal Quantum Computation with AKLT States
and Spectral Gap of AKLT Models

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Workshop on Statistical Physics of Quantum Matter, Taipei, July 2013

Outline

I. Introduction: quantum computation by local measurement
--- cluster-state (one-way) quantum computer

II. QC on AKLT (Affleck-Kennedy-Lieb-Tasaki) states

1) 1D: spin-1 AKLT chain
2) 2D: spin-3/2 AKLT state on honeycomb, square-octagon, cross & star lattices

III. Finite gap of 2D AKLT Hamiltonians ---numerical evidence

IV. Summary and outlook

Collaborators:

Ian Affleck (UBC)

Robert Raussendorf (UBC)

Artur Garcia (Stony Brook)

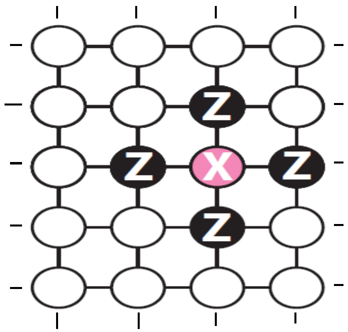
Valentin Murg (Vienna)

One-Way Quantum Computation: by Local Measurement

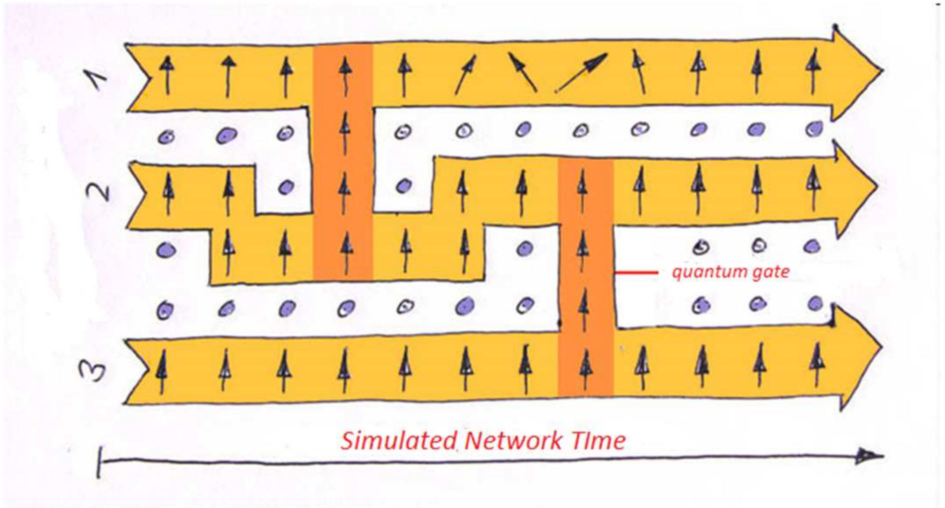
❑ Single-qubit measurements on the 2D cluster state gives rise to universal quantum computation (QC)

[*Raussendorf &Briegel*, PRL01']

➢ 2D cluster state



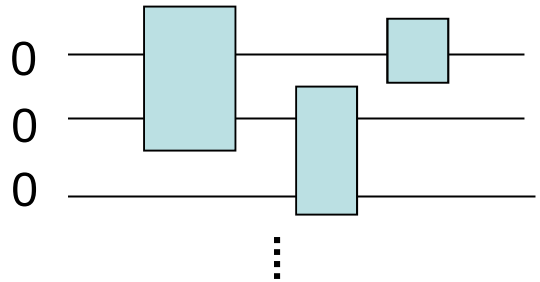
QC = pattern of measurement



➢ Key points:

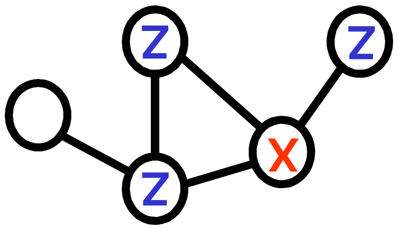
✓ Equivalent to circuit model:

✓ Universal gates can be implemented



Cluster state and graph state

- Graph states: defined on any graph [*Hein, Eisert & Briegel* 04']



- ➢ Via stabilizer generators:

$$K_v|G\rangle = |G\rangle, \ \forall \text{ vertices } v$$ [These eqs. uniquely define |G>.]

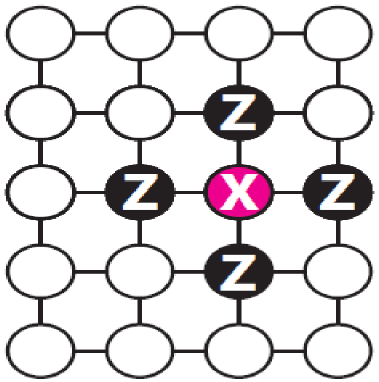
$$K_v = X_v \bigotimes_{u \in \mathrm{Nb}(v)} Z_u$$ (X,Y,Z: Pauli matrices)

- ➢ Via controlled-Z gates: $$|G\rangle = \prod_{\text{edge } \langle i,j \rangle} CZ_{ij} \big(|+\rangle|+\rangle \cdots |+\rangle\big)$$

$$|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$$

- Cluster states: special case of graph states on regular lattices, e.g. square

[*Raussendorf &Briegel* 01']



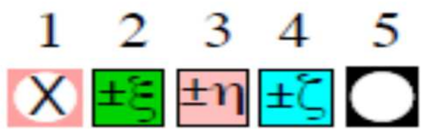
$$CZ \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Universal gate set: Lego pieces for QC

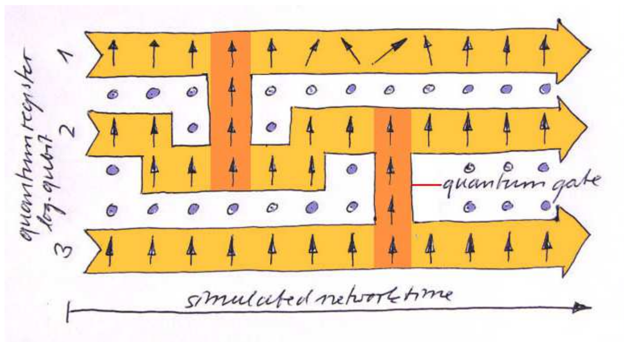
[*Raussendorf &Briegel* PRL 01']

- Cluster-state QC = a set of measurement patterns

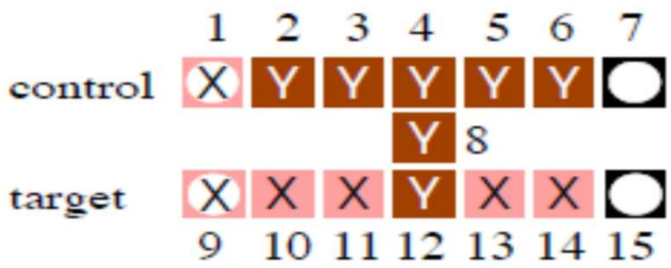
1. Can isolate wires for single-qubit gates



general rotation



2. CNOT gate via entanglement between wires



CNOT-gate

$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Search for universal resource states

- Can other states beyond the 2D cluster state be used for measurement-based quantum computation?

- Other known examples:
  - Any other 2D *graph* states on regular lattices (≡cluster states): triangular, honeycomb, kagome, etc. [*Van den Nest et al.* '06]
  - MPS & PEPS framework: alternative view & further examples

[*Verstraete & Cirac* '04] [*Gross & Eisert* '07, *Gross, Eisert, Schuch & Perez-Garcia* '10]

- Can universal resource states be unique ground state?

➔ Create resources by cooling (if Hamiltonian is gapped)!

➔ Desire simple and short-ranged (nearest nbr) 2-body Hamiltonians

Cluster states: not unique ground state of two-body Hamiltonians

[*Nielsen* '04]

We will focus on the family of Affleck-Kennedy-Lieb-Tasaki (AKLT) states

➔ Unique ground states of short-ranged (nearest nbr) 2-body Hamiltonians

➔ For certain cases (mostly 1D chains), existence of a finite gap above the ground state can be proved

➔ But can they be useful for quantum computation?

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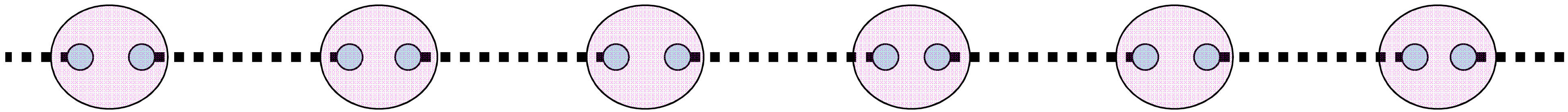
1D AKLT state

[*AKLT* '87,'88]

- Spin-1 chain: two virtual qubits per site



$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

Project into symmetric subspace of two spin-1/2 (qubits)

$$\begin{cases} |00\rangle & \Rightarrow |1,1\rangle \\ |11\rangle & \Rightarrow |1,-1\rangle \\ (|01\rangle + |10\rangle)/\sqrt{2} & \Rightarrow |1,0\rangle \end{cases}$$

singlet

$$|01\rangle - |10\rangle = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

✓ Can realize rotation on one logical qubit by measurement

[*Gross & Eisert, PRL* '07] [*Brennen & Miyake, PRL* '09]

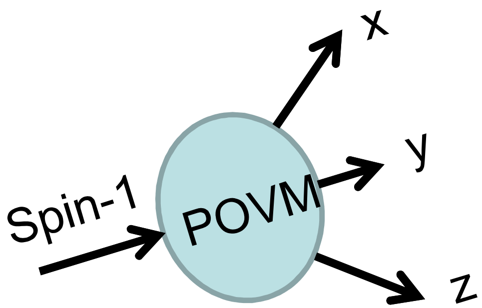
➢ One reason: 1D AKLT state can be converted to 1D cluster state by local measurement (and 1D cluster state can realize 1-qubit rotation)

1D AKLT state ➔ cluster state

- Our approach uses a POVM: $\sum_{\alpha=x,y,z} F_\alpha^\dagger F_\alpha = I_{S=1}$ (outcome: x, y, z)

$$
\begin{aligned}
F_x &= \sqrt{\frac{1}{2}}(|++\rangle\langle++|+|--\rangle\langle--|) \\
F_y &= \sqrt{\frac{1}{2}}(|i,i\rangle\langle i,i|+|-i,-i\rangle\langle -i,-i|) \\
F_z &= \sqrt{\frac{1}{2}}(|00\rangle\langle 00|+|11\rangle\langle 11|)
\end{aligned}
$$

$$|\pm i\rangle \equiv (|0\rangle \pm i|1\rangle)/\sqrt{2}$$

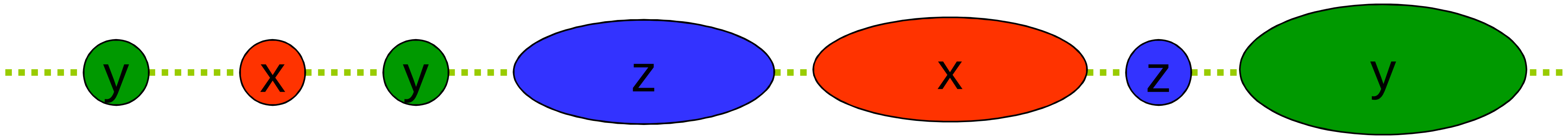


Any outcome preserves a two-dimensional subspace

$$|\Phi\rangle \longrightarrow F_\alpha|\Phi\rangle$$

[Wei, Affleck & Raussendorf '12]

➔ gives rise to a cluster state (a logical qubit is a **domain** of connected sites with same outcome)

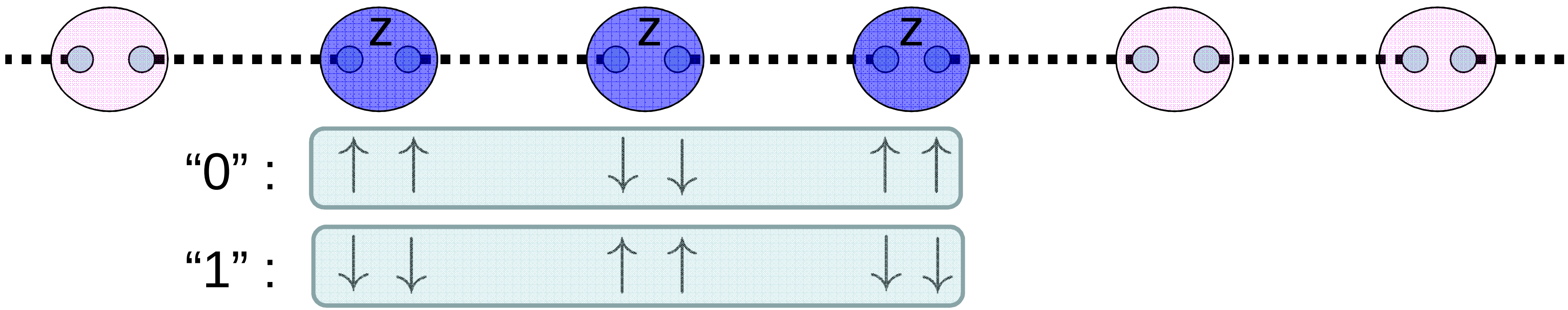


➔ In a large system, cluster state has length 2/3 of AKLT

Remarks on two key points:

(1) A domain is formed by merging connected sites with same outcome and is a logical qubit:

- Anti-ferromagnetic properties from singlets



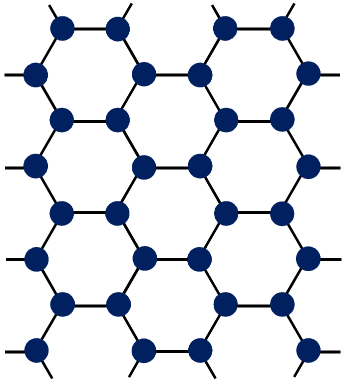
(2) No leakage out of qubit encoding due to

$$\sum_{\alpha=x,y,z} F_\alpha^\dagger F_\alpha = I_{S=1}$$ (probability adds up to 1)

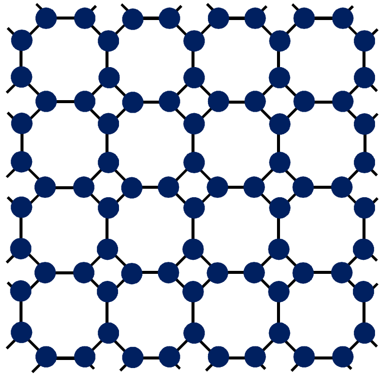
- Random outcome x, y, or z indicates quantization axis

1D AKLT state can only support 1-qubit rotation, not universal QC; What about 2D AKLT states?

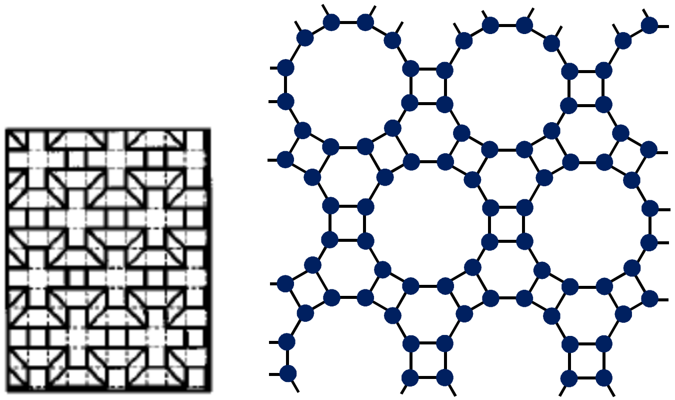
(a) honeycomb



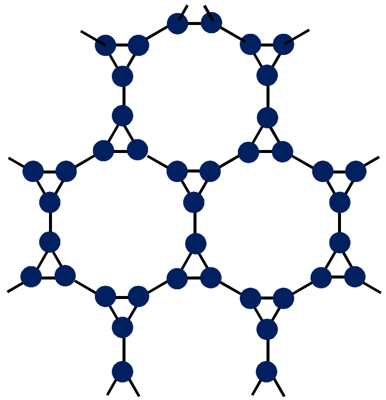
(b) square-octagon



(c) ‘cross’



(d) ‘star’



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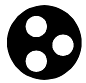
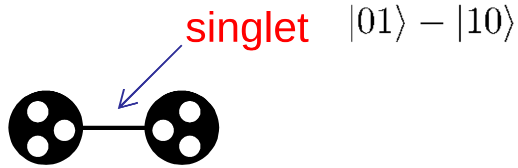
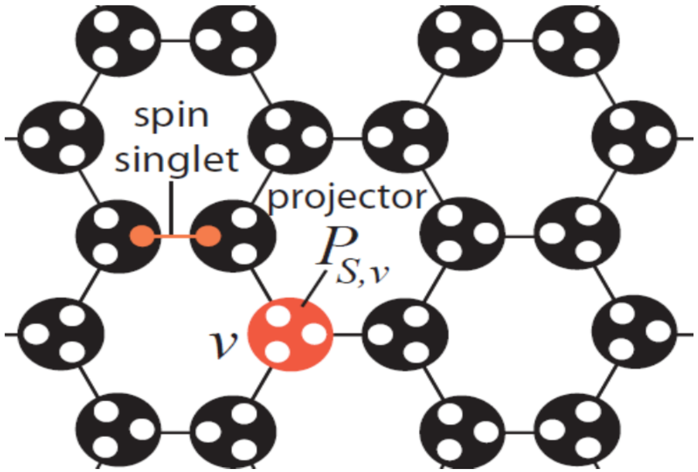
Spin-3/2 AKLT state on honeycomb

- Each site contains three virtual qubits 



- Two virtual qubits on an edge form a singlet



Spin 3/2 and three virtual qubits

- Addition of angular momenta of 3 spin-1/2's

$$\frac{1}{2}\otimes\frac{1}{2}=0\oplus 1 \qquad \frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2}=\frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}$$

Symmetric subspace

- The four basis states in the symmetric subspace

$$|000\rangle \leftrightarrow \left|\frac{3}{2},\frac{3}{2}\right\rangle$$

$$|W\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle+|010\rangle+|100\rangle) \leftrightarrow \left|\frac{3}{2},\frac{1}{2}\right\rangle$$

$$|\overline{W}\rangle \equiv \frac{1}{\sqrt{3}}(|110\rangle+|101\rangle+|011\rangle) \leftrightarrow \left|\frac{3}{2},-\frac{1}{2}\right\rangle$$

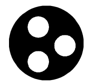
$$|111\rangle \leftrightarrow \left|\frac{3}{2},-\frac{3}{2}\right\rangle$$

Effective 2 levels of a qubit

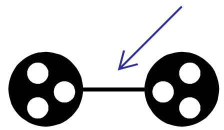
- Projector onto symmetric subspace

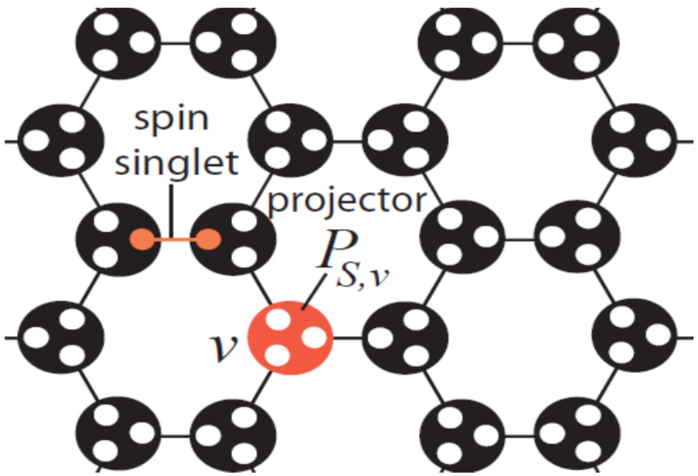
$$P_{S,v} = |000\rangle\langle 000| + |111\rangle\langle 111| + |W\rangle\langle W| + |\overline{W}\rangle\langle\overline{W}| \leftrightarrow I_{3/2}$$

Spin-3/2 AKLT state on honeycomb

- Each site contains three virtual qubits 

singlet $|01\rangle - |10\rangle$

- Two virtual qubits on an edge form a singlet 



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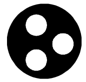
- Each site contains three virtual qubits 

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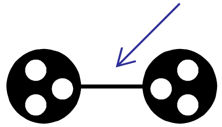
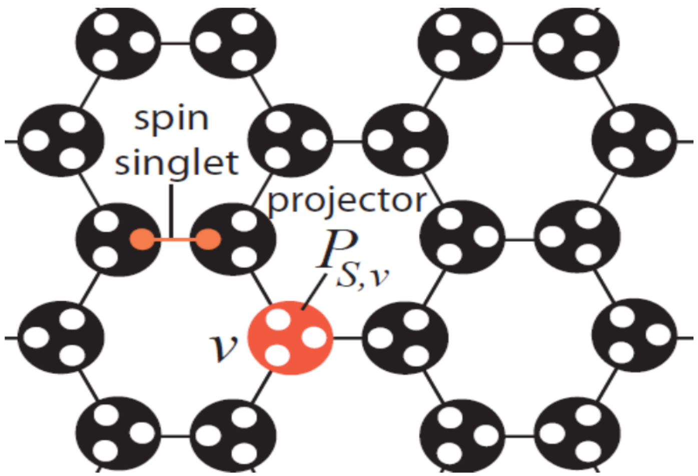
- Two virtual qubits on an edge form a singlet 

- Projection ($P_{S,v}$) onto symmetric subspace of 3 qubits at each site & relabeling with spin-3/2 (four-level) states



$$P_{S,v} = |000\rangle\langle 000| + |111\rangle\langle 111| + |W\rangle\langle W| + |\overline{W}\rangle\langle\overline{W}|$$

$$|000\rangle \leftrightarrow \left|\frac{3}{2},\frac{3}{2}\right\rangle \qquad |111\rangle \leftrightarrow \left|\frac{3}{2},-\frac{3}{2}\right\rangle$$

$$|W\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \leftrightarrow \left|\frac{3}{2},\frac{1}{2}\right\rangle$$

$$|\overline{W}\rangle \equiv \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle) \leftrightarrow \left|\frac{3}{2},-\frac{1}{2}\right\rangle$$

Convert to graph states via POVM

$$F_{v,z} = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_z + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_z\right) = \frac{1}{\sqrt{6}}\left(S_z^2 - \frac{1}{4}\right)$$

$$F_{v,x} = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_x + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_x\right) = \frac{1}{\sqrt{6}}\left(S_x^2 - \frac{1}{4}\right)$$

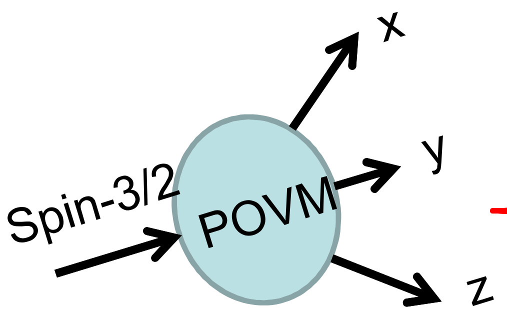
$$F_{v,y} = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_y + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_y\right) = \frac{1}{\sqrt{6}}\left(S_y^2 - \frac{1}{4}\right)$$

[*Wei*,*Affleck & Raussendorf* '11; *Miyake* '11]

v: site index

➔ Three elements satisfy: $F_{v,x}^\dagger F_{v,x} + F_{v,y}^\dagger F_{v,y} + F_{v,z}^\dagger F_{v,z} = I_v$

- POVM outcome (*x*,*y*, or *z*) is random (a_v ={x,y,z} ϵ *A* for all sites v)



➔ effective 2-level system (logical qubit = domain)

$$\left|\frac{3}{2}\right\rangle_{a_v} \leftrightarrow |000\rangle, \quad \left|-\frac{3}{2}\right\rangle_{a_v} \leftrightarrow |111\rangle$$

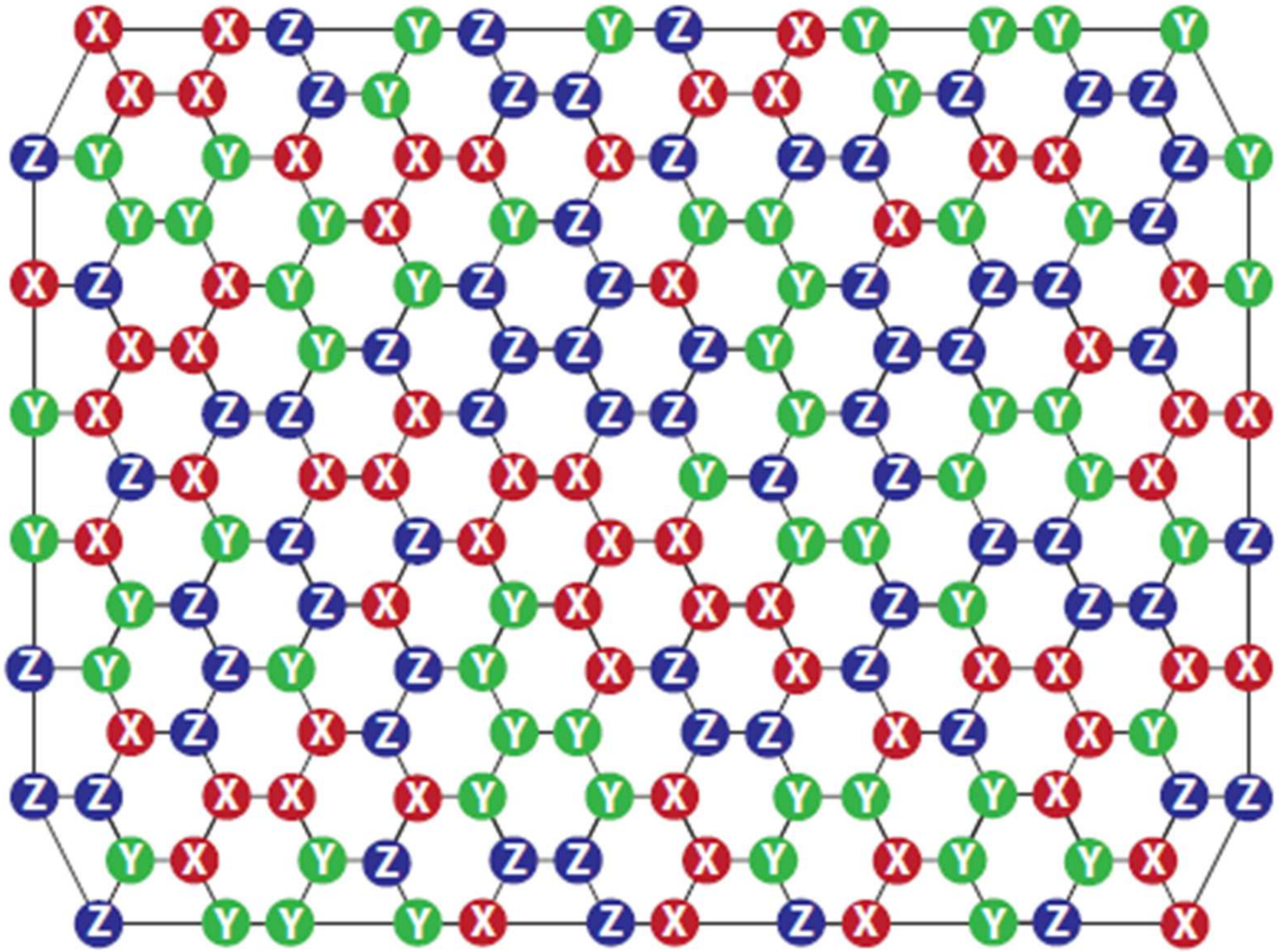
➔ a_v : new quantization axis

$$\overline{Z} \equiv \left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_{a_v} - \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_{a_v} \qquad \overline{X} \equiv \left|\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_{a_v} + \left|-\frac{3}{2}\right\rangle\left\langle+\frac{3}{2}\right|_{a_v}$$

➔ state becomes $|\Phi\rangle \longrightarrow F_{v,a_v}|\Phi\rangle$

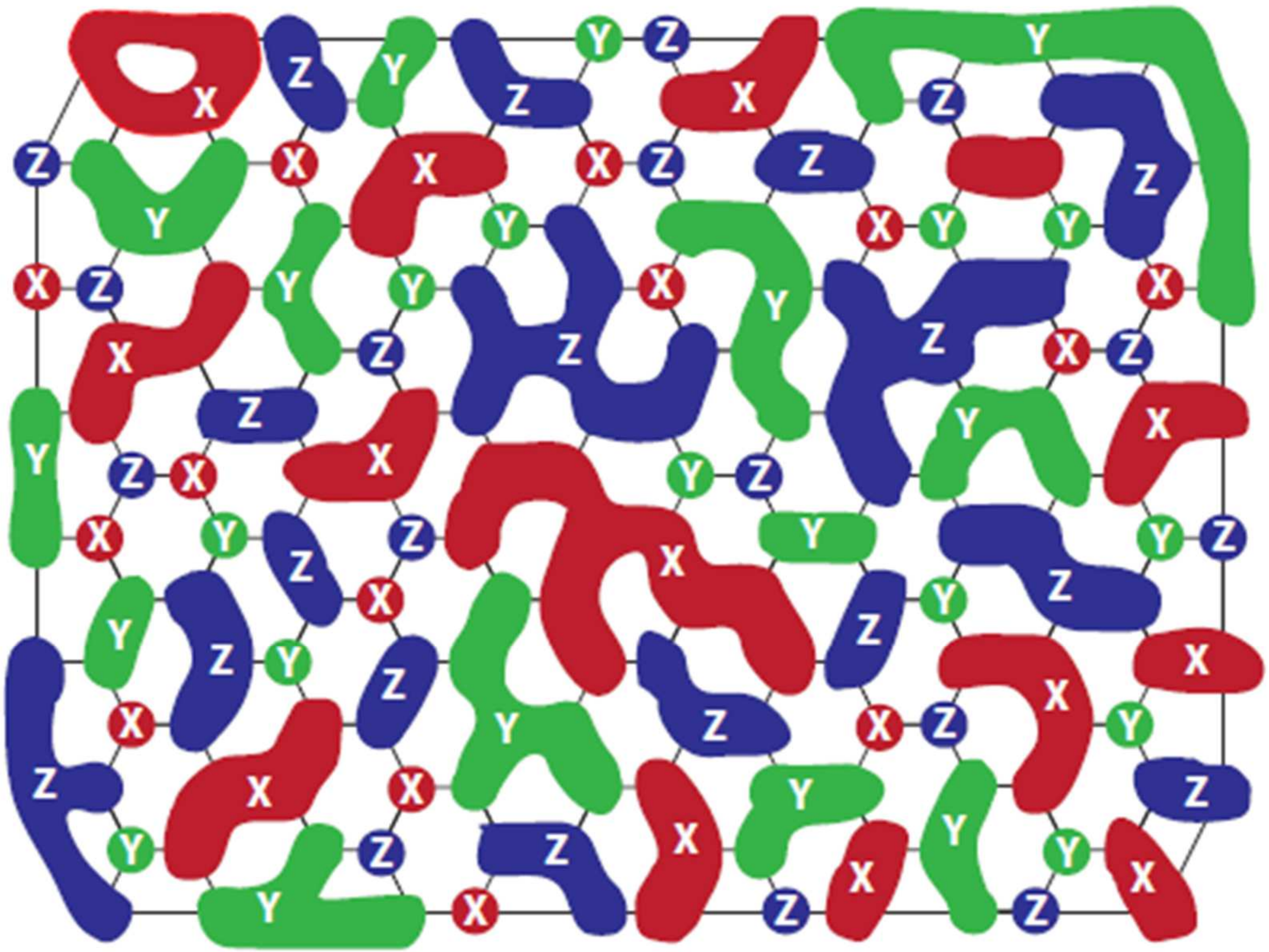
AKLT on honeycomb

1. Random x, y, z outcomes



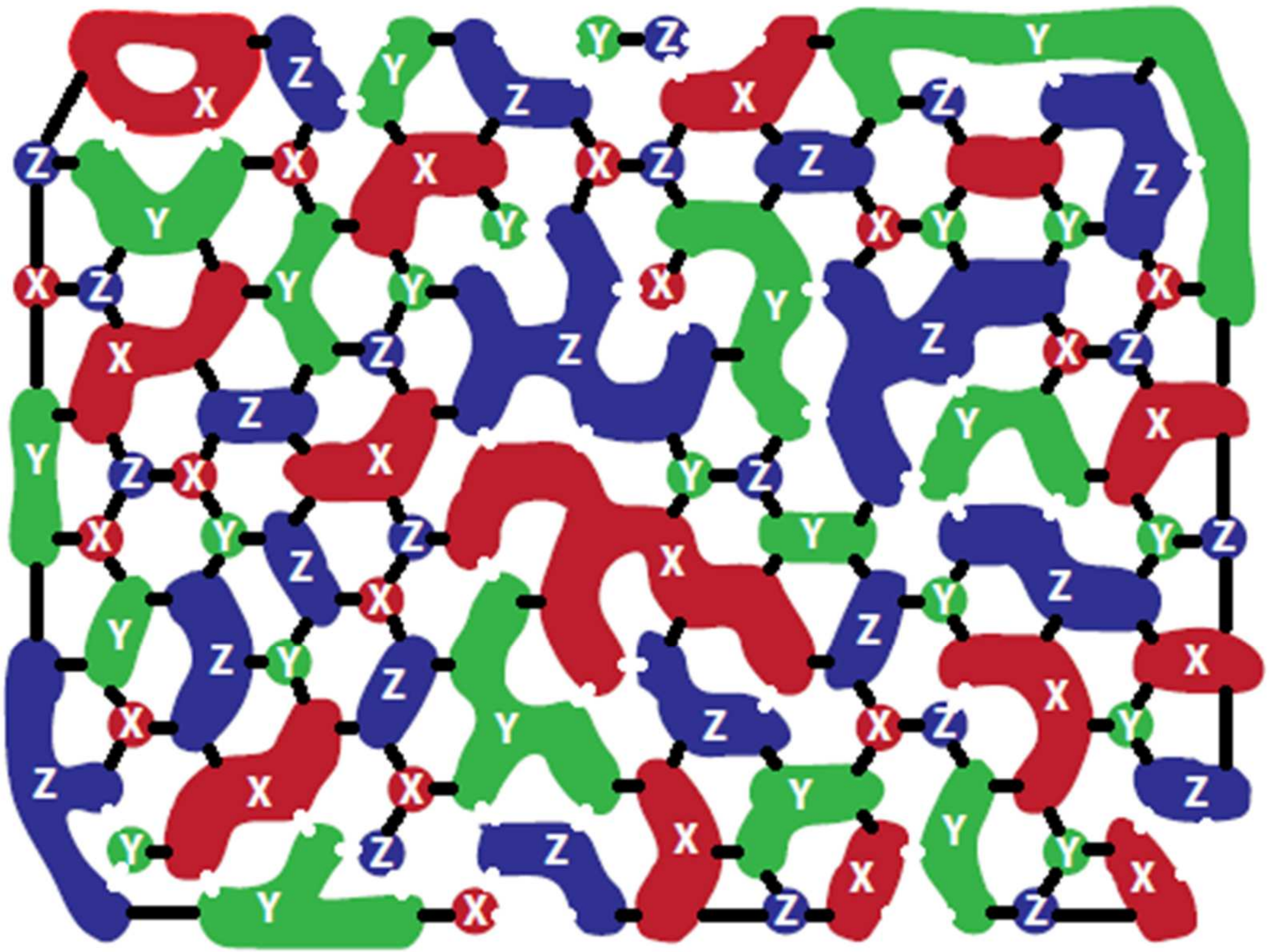
AKLT on honeycomb

2. Merge sites to domains
(1 domain= 1 logical qubit)



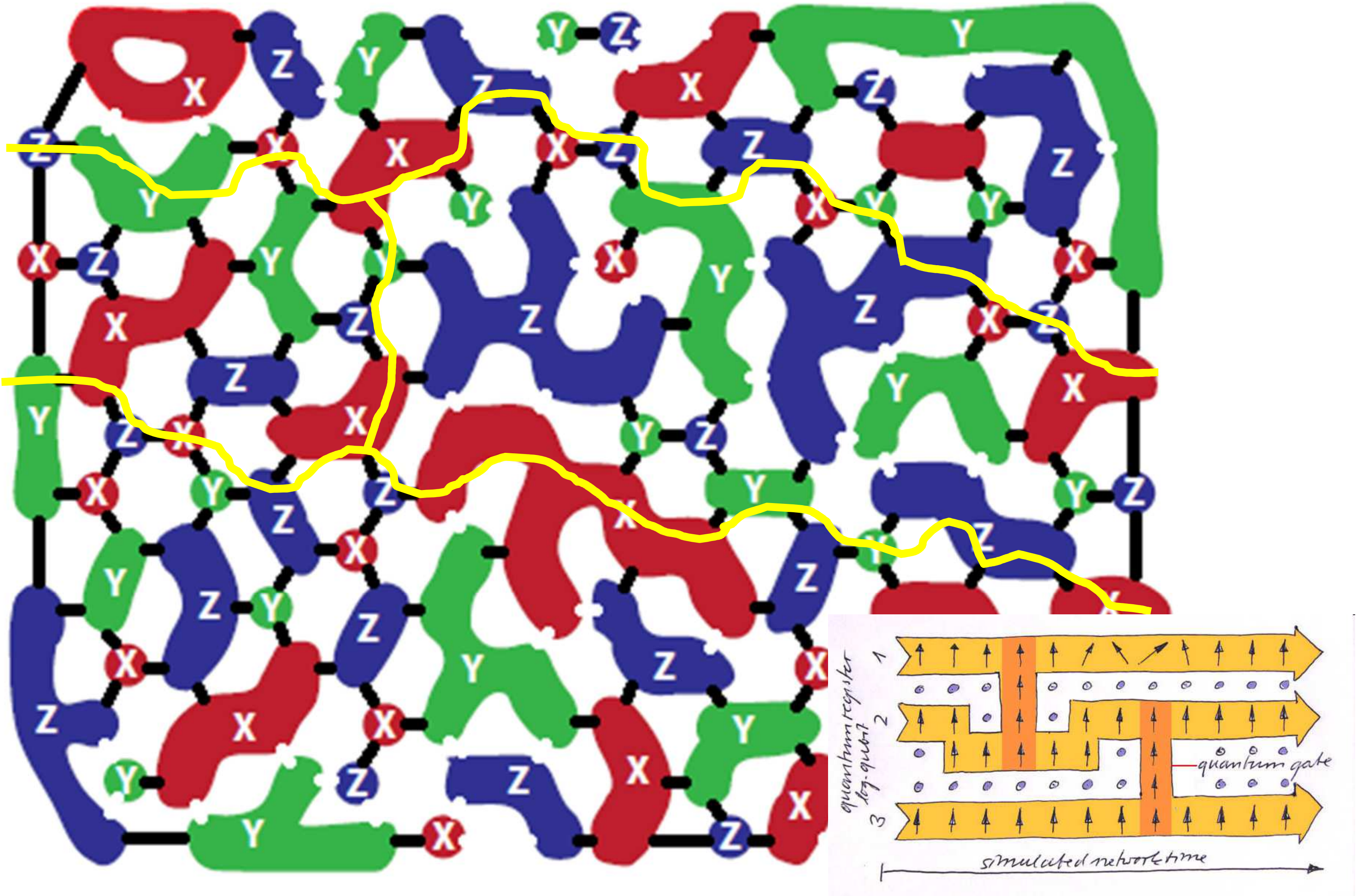
AKLT on honeycomb

3. Even # edges = 0 edge
   Odd # edges = 1 edge
   (New feature in 2D)



$$\sigma_z^2 = I$$

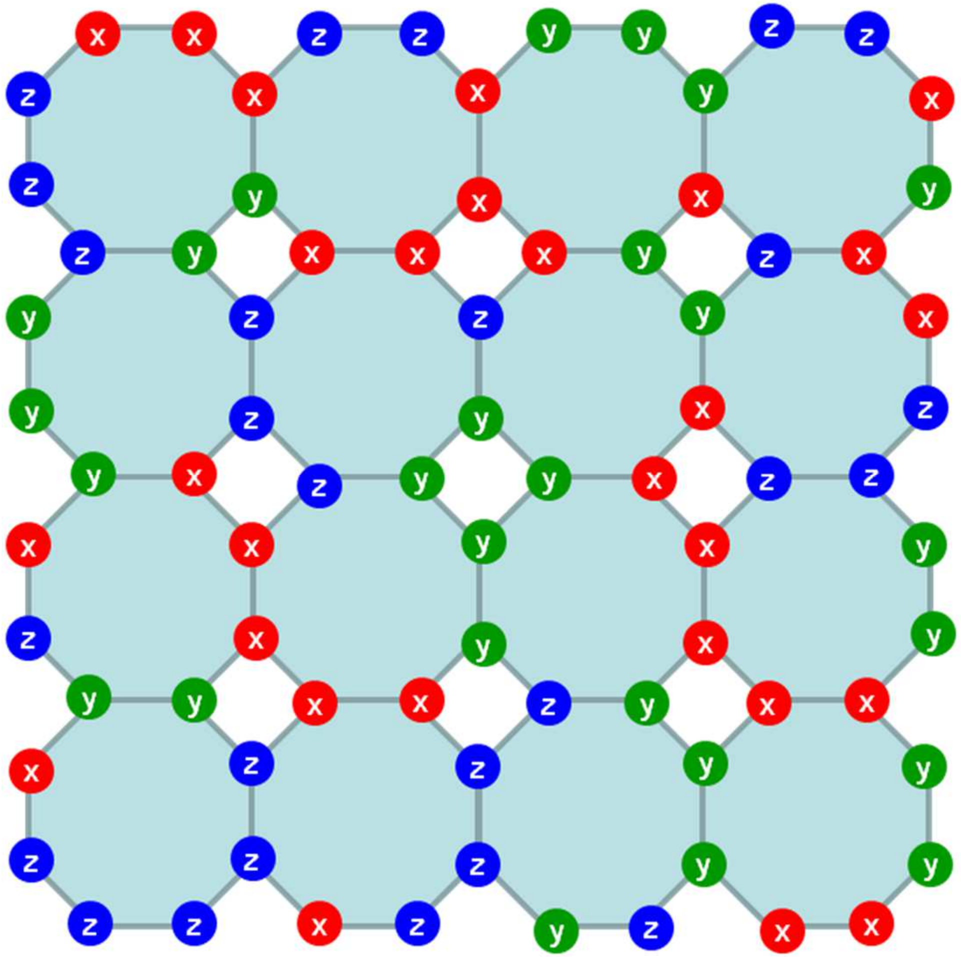
Quantum computation can be implemented on such a (random) graph state



- Sufficient number of wires if graph is in supercritical phase (percolation)

AKLT on square-octagon

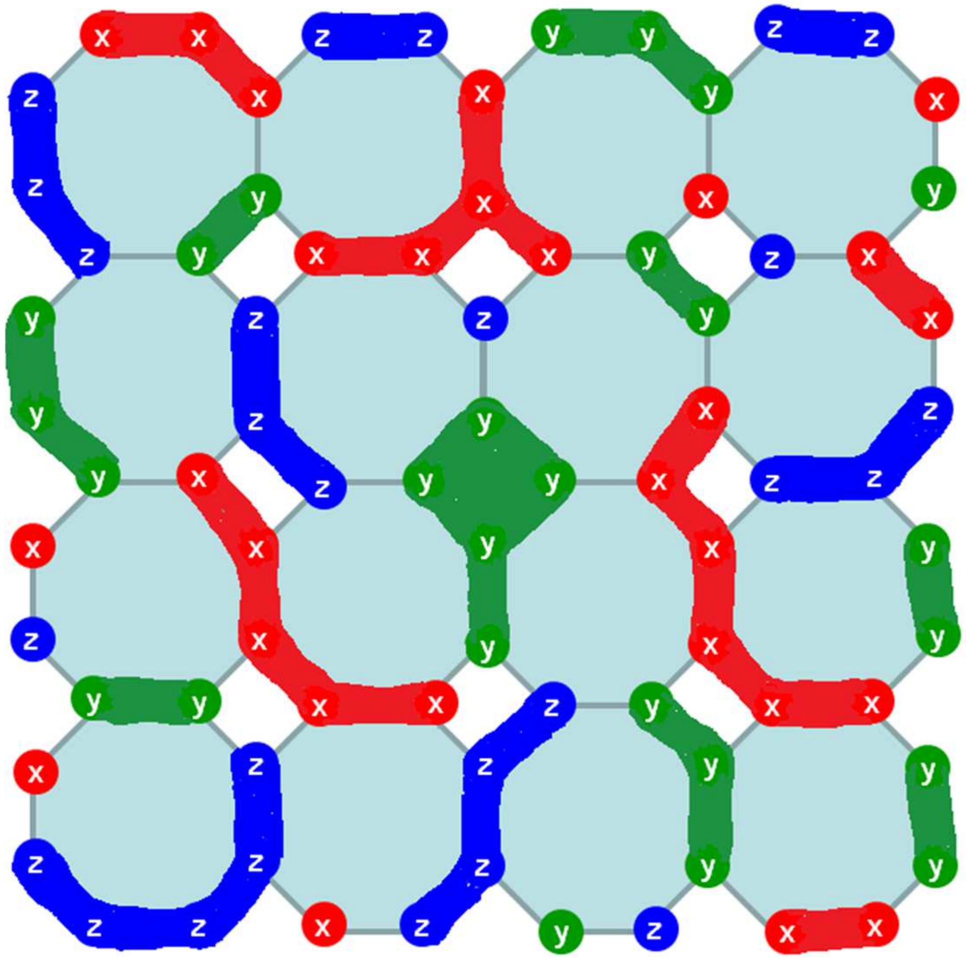
- Follow the same procedure



Bond Percolation
Threshold ≈ 0.6768
> 2/3

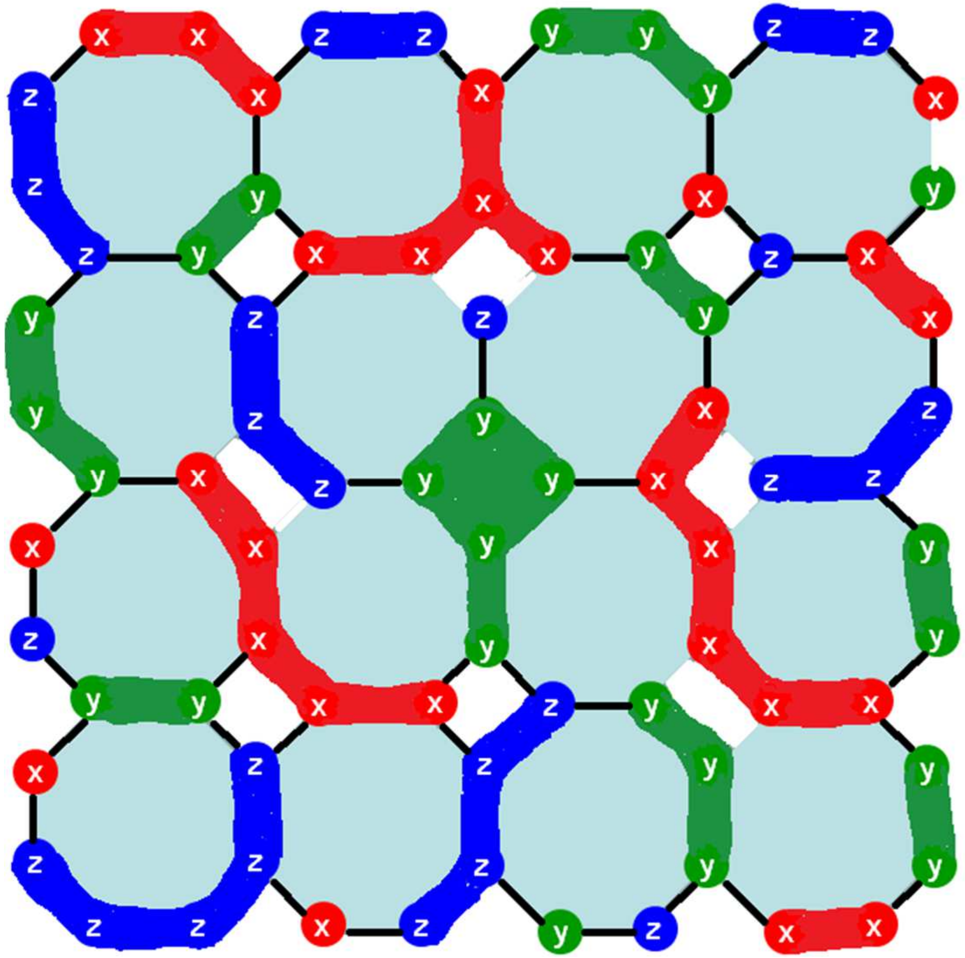
Merge sites to domains

- Neighboring sites with same POVM outcome
  ➔ one domain = one qubit



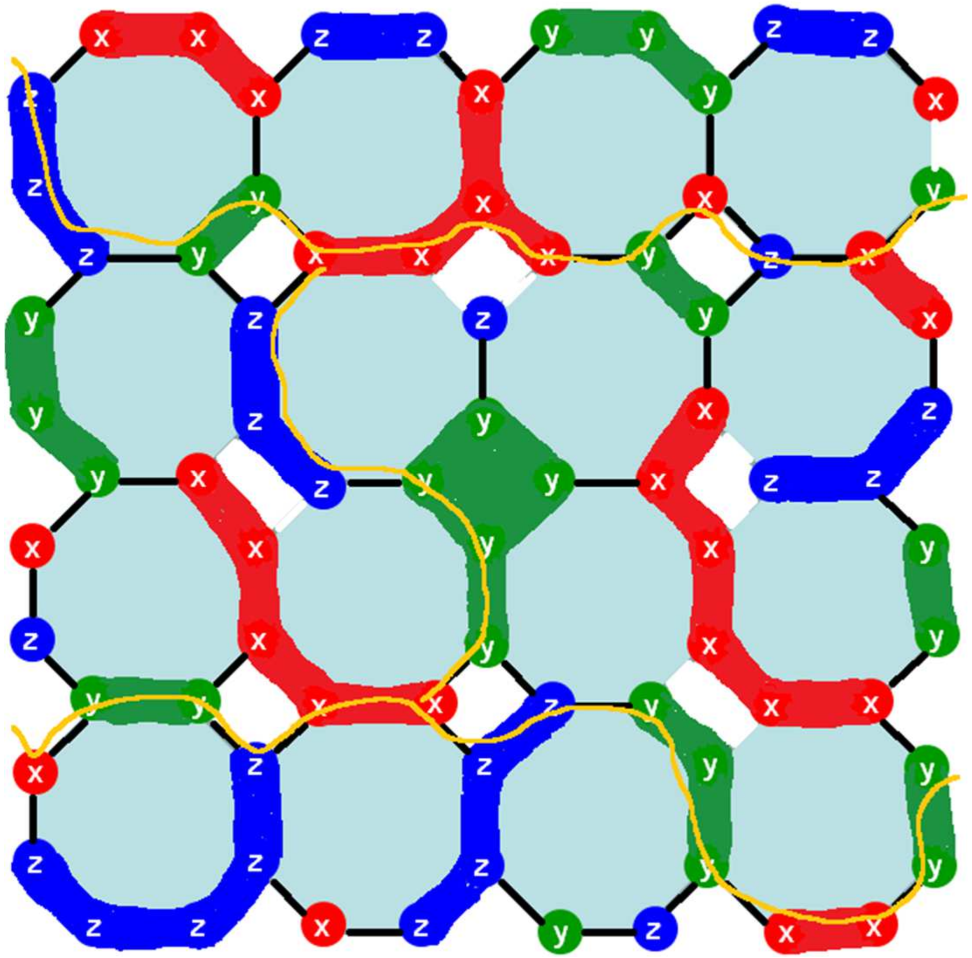
Graph state: the graph

- Two domains connected by even edges = no edge
  odd edges = 1 edge



QC on the new graph

- Identify new “backbone” (may not exist on original graph)

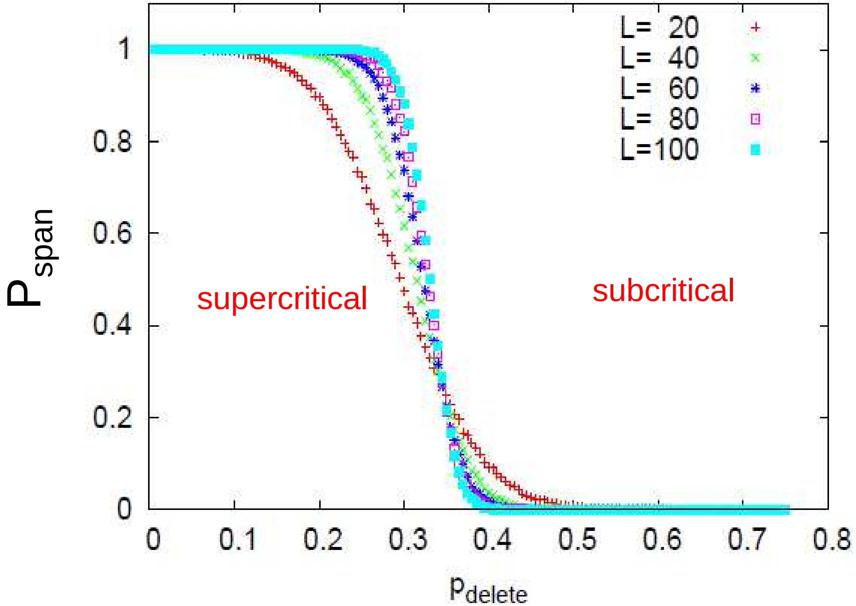


Robustness: finite percolation threshold

❑ Typical graphs are in percolated (or supercritical) phase

Site percolation by deletion (Honeycomb)

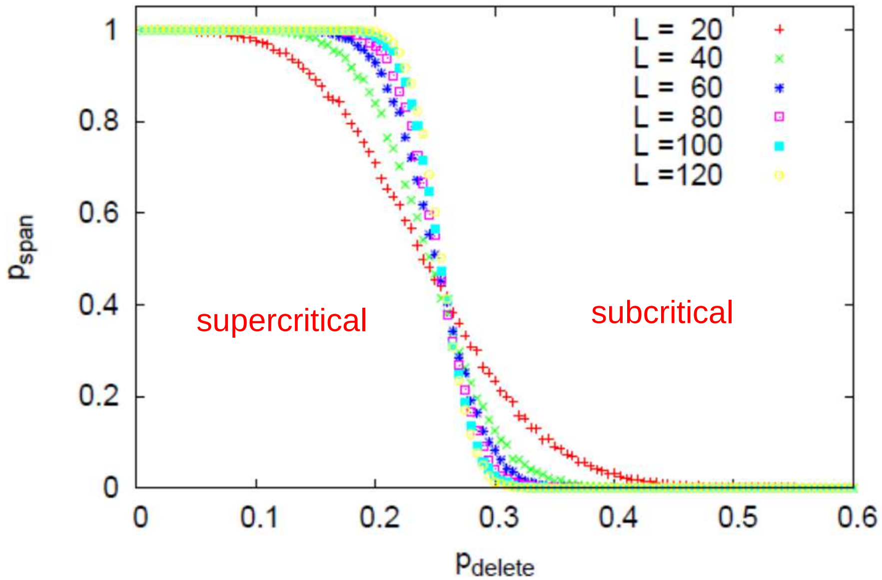
[*Wei*, *Affleck & Raussendorf* '11]



➔ Threshold = 1- P_{delete}* ≈1-0.33=0.67

Site percolation by deletion (Square-octagon)

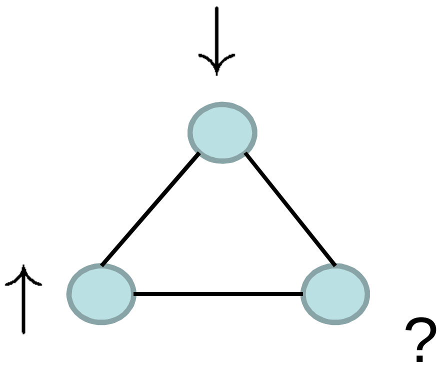
[*Wei* '13]



➔ threshold ≈1-0.26=0.74

- ➔ Sufficient (macroscopic) number of traversing paths exist (supercritical)
- ➔ These AKLT states (also that on 'cross') are universal for QC

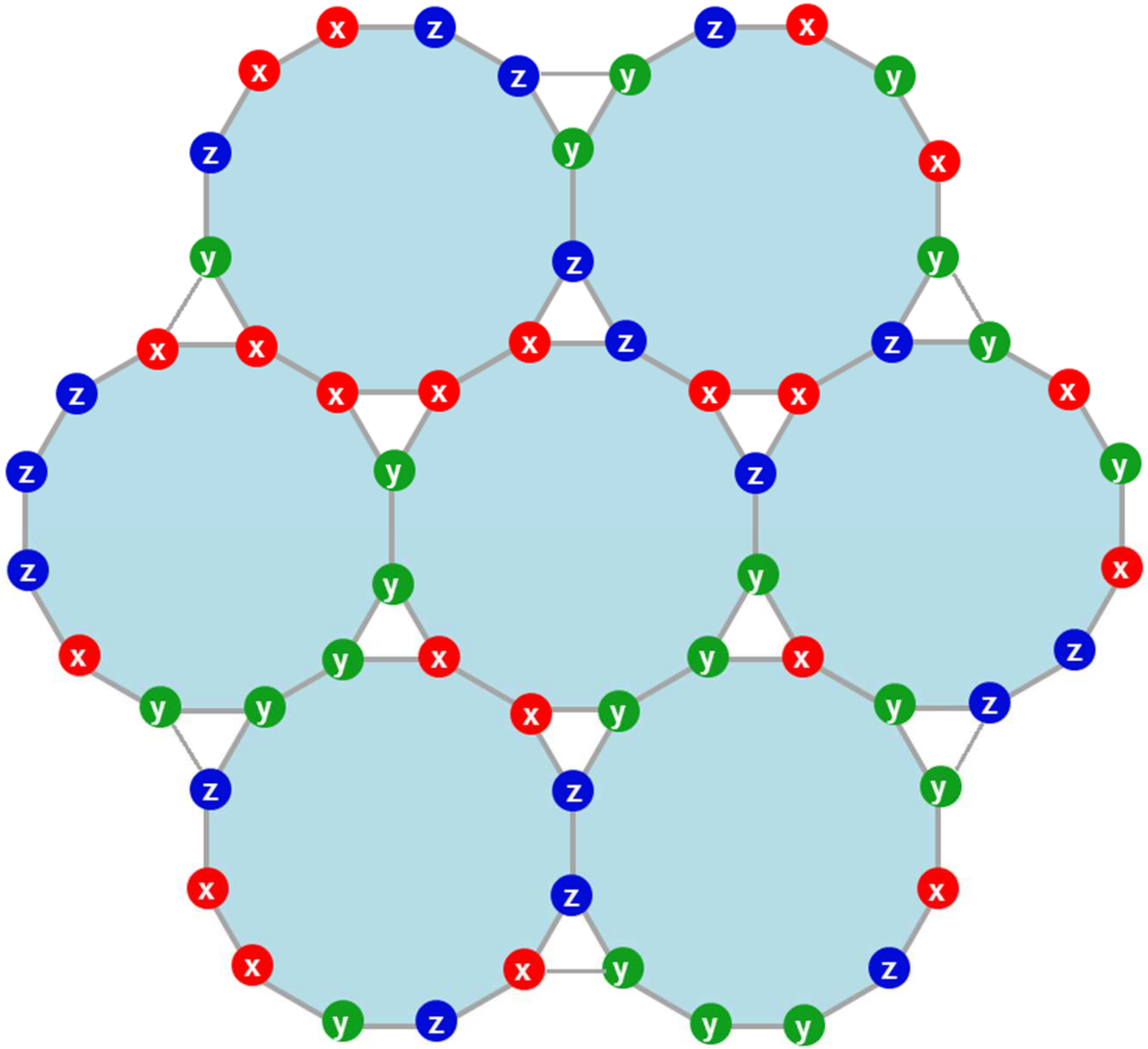
However, the AKLT state on the star lattice is NOT universal, due to **frustration**!



➔ Cannot have POVM outcome xxx, yyy or zzz on a triangle

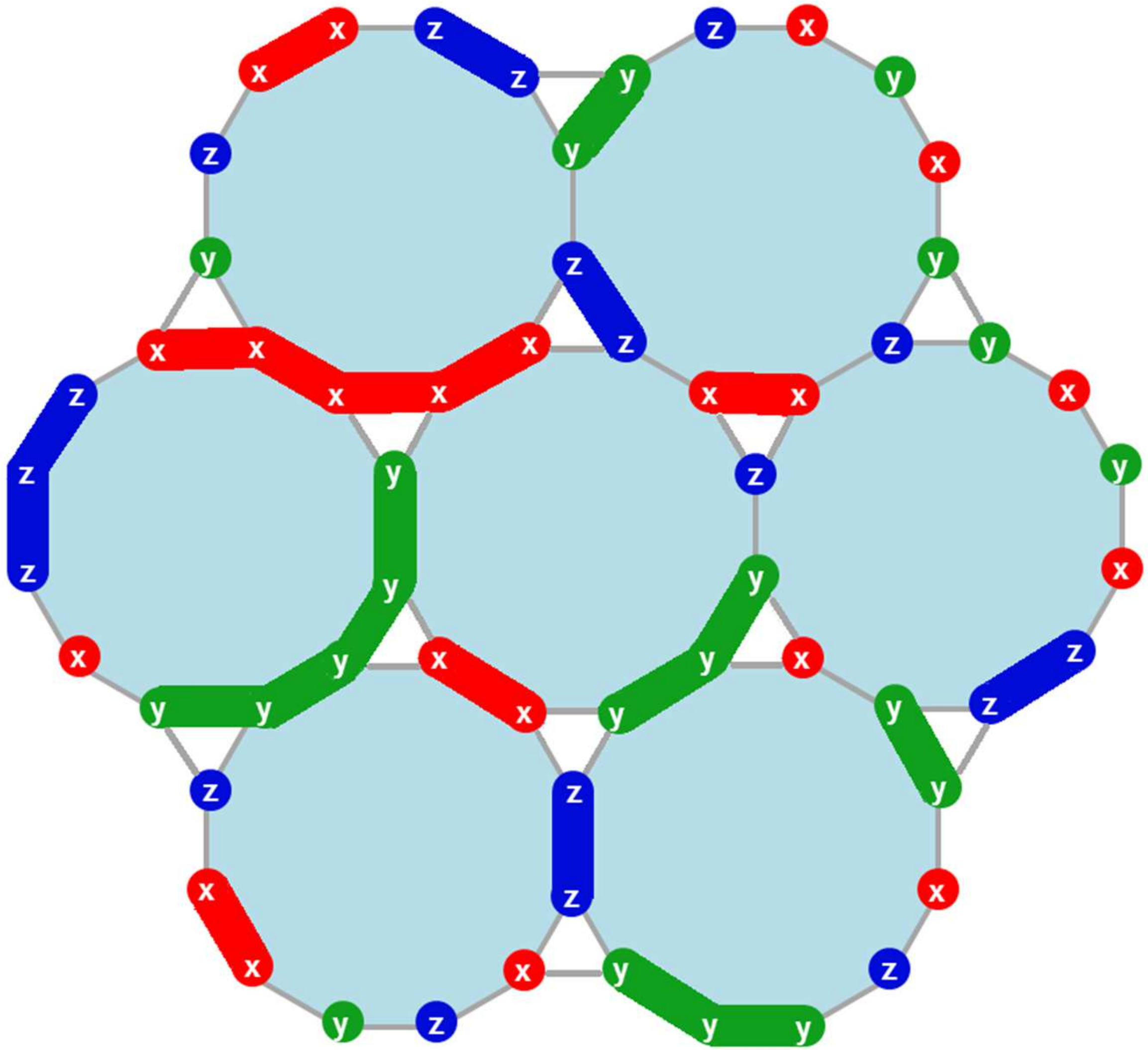
AKLT on star lattice

1. Random x, y, z outcomes



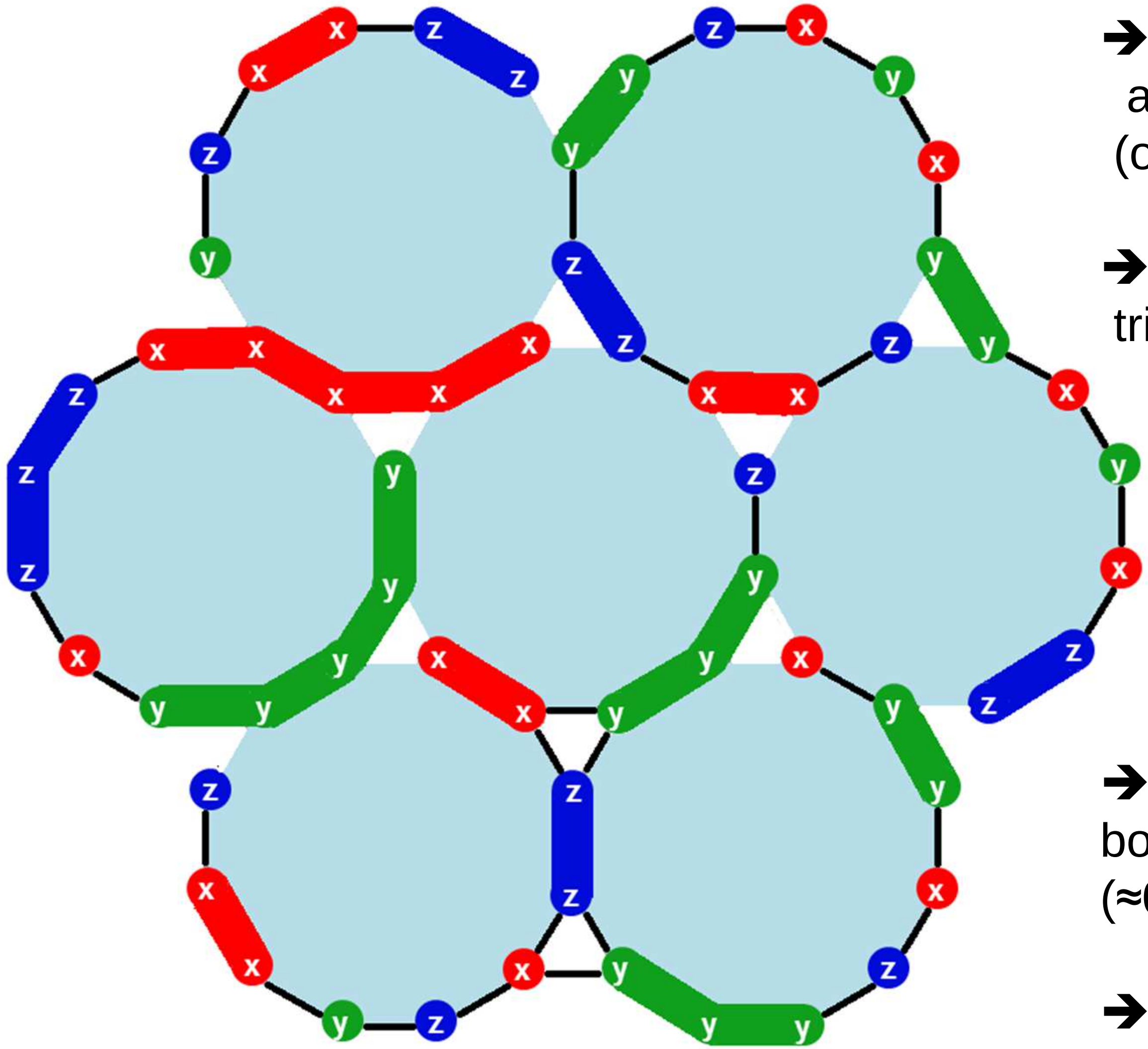
AKLT on star lattice

2. Merge sites to domains



AKLT on star lattice

3. Edge modulo 2 operation



➔ Edges in triangles are removed with 50% (occupied with 50%)

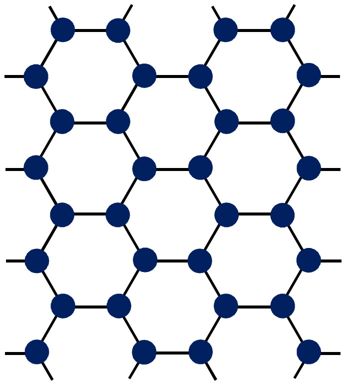
➔ Edges connecting triangles never removed

➔ 50% is smaller than bond percolation threshold (≈0.5244) of Kagome

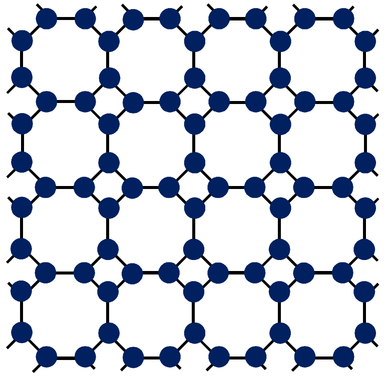
➔ No connected path
➔ AKLT not universal

AKLT states: universal resource or not?

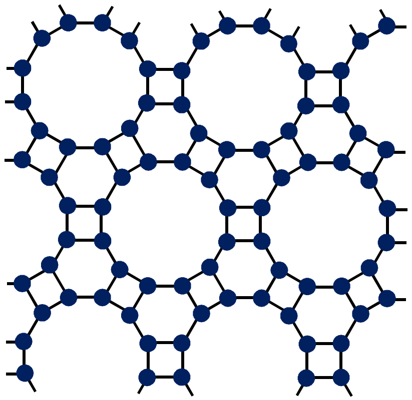

honeycomb



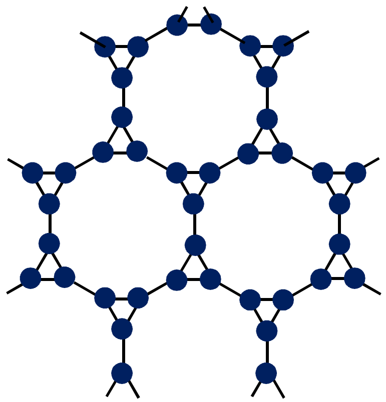

square-octagon




‘cross’




‘star’



AKLT state on square lattice?

- Whether such spin-2 state is universal remains open
  - Technical problem: trivial extension of POVM does NOT work!

$$
\begin{aligned}
F_z &= \left|2\right\rangle\left\langle 2\right|_z + \left|-2\right\rangle\left\langle -2\right|_z \\
F_x &= \left|2\right\rangle\left\langle 2\right|_x + \left|-2\right\rangle\left\langle -2\right|_x \\
F_y &= \left|2\right\rangle\left\langle 2\right|_y + \left|-2\right\rangle\left\langle -2\right|_y
\end{aligned}
$$

$$F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z \neq c \cdot I$$

➔ Leakage out of logical subspace (error)

Outline

I. Introduction: quantum computation by local measurement

II. QC on AKLT (Affleck-Kennedy-Lieb-Tasaki) states

1) 1D: spin-1 AKLT chain

2) 2D: spin-3/2 AKLT state on honeycomb, square-octagon, cross & star lattices

III. Finite gap of 2D AKLT Hamiltonians ---numerical evidence

IV. Summary and outlook

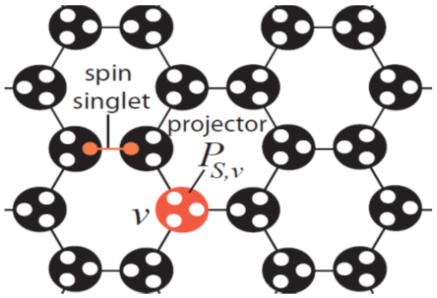
Finite gap of spin-3/2 AKLT model?

- Hamiltonian

[*AKLT* '87,'88]

$$H_{\text{AKLT}} = \sum_{\text{edge } \langle i,j \rangle} \hat{P}_{i,j}^{(S=3)} = \frac{27}{160} \sum_{\text{edge } \langle i,j \rangle} \left[\vec{S}_i \cdot \vec{S}_j + \frac{116}{243} (\vec{S}_i \cdot \vec{S}_j)^2 + \frac{16}{243} (\vec{S}_i \cdot \vec{S}_j)^3 + \frac{55}{108} \right]$$

- Known to have exponential decaying correlation functions, but NOT a proof of gap



- We use tensor network methods to show the existence of gap and its value

  - See Artur Garica's poster for details

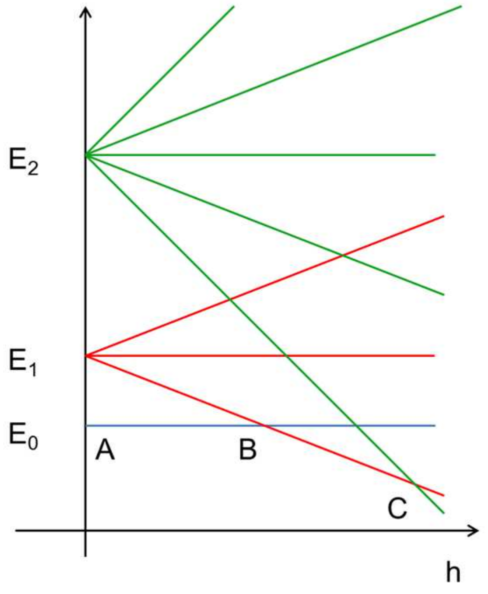
Inferring gap of AKLT models

- Ground state is a spin singlet state;
  eigenstates characterized by total |S, Sz ›

- By applying an external field, can probe the gap

$$H = H_{\mathrm{AKLT}} + h \sum_i S_i^z$$

➢ Schematic energy response

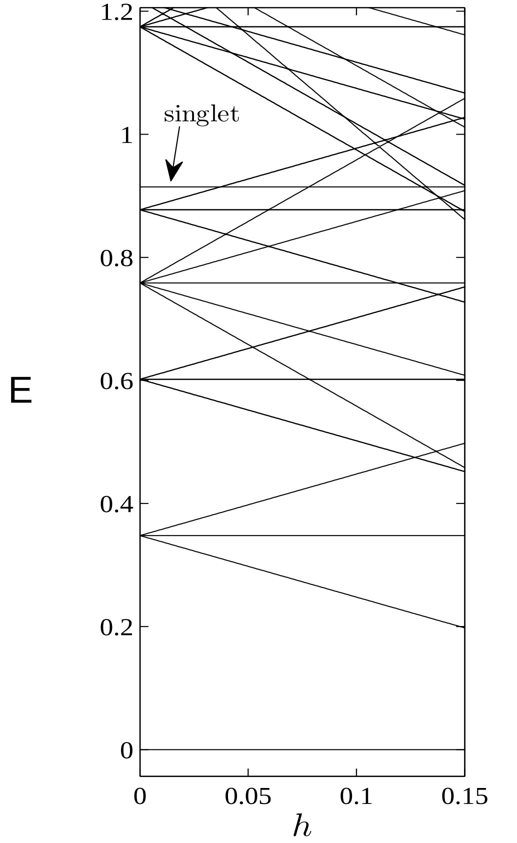


❖ A, B, C, … traces lowest energy of H

❖ First cross and the slope
➔ infer E_1 - E_0

❖ Slope = Magnetization Plateau ➔ finite gap

➢ 1D AKLT with N=8

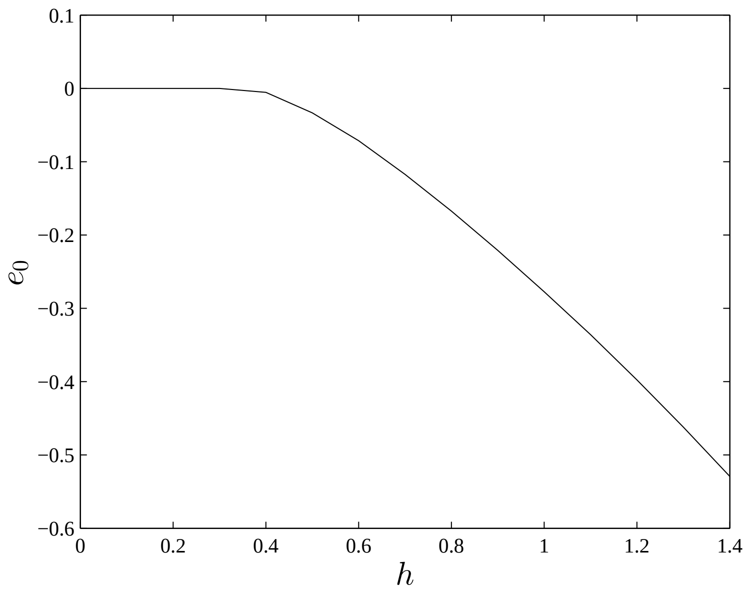


1D spin-1 AKLT model

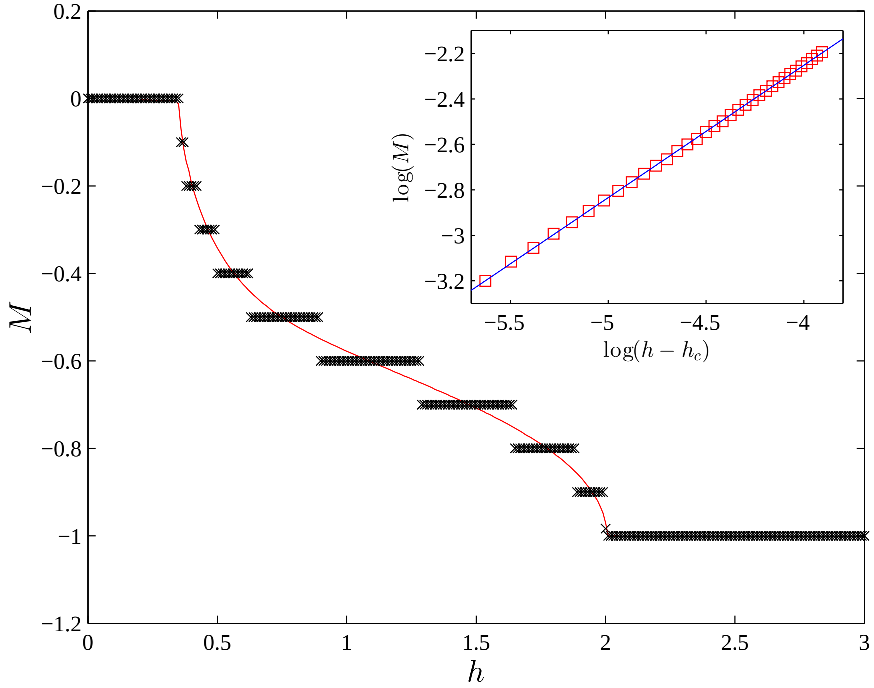
- Hamiltonian: $$H = H_{\rm AKLT} + h \sum_i S_i^z$$

$$H_{\rm AKLT} = \frac{1}{2}\sum_i \left[\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_{i+1})^2 + \frac{2}{3}\right]$$

Energy per spin



Magnetic moment per spin



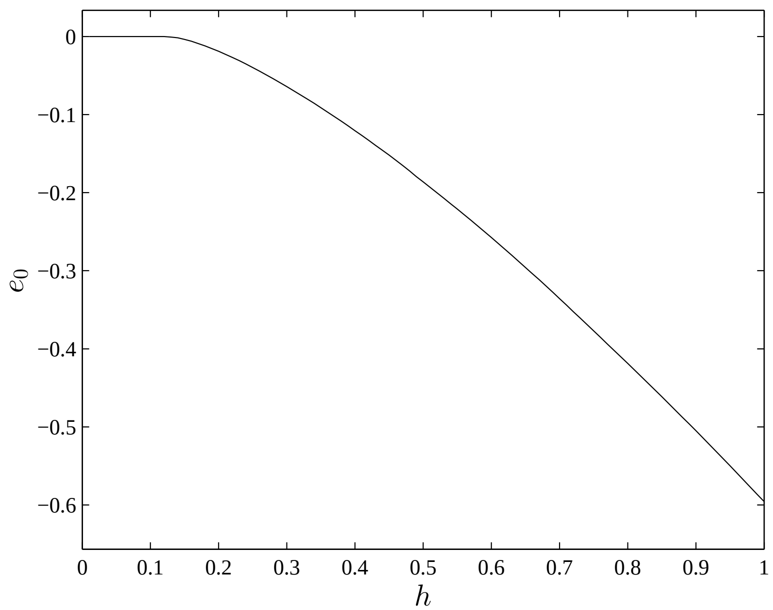
➔ Gap $\Delta \approx 0.350$

2D spin-3/2 AKLT on honeycomb

- Hamiltonian: $H = H_{\mathrm{AKLT}} + h \sum_i S_i^z$

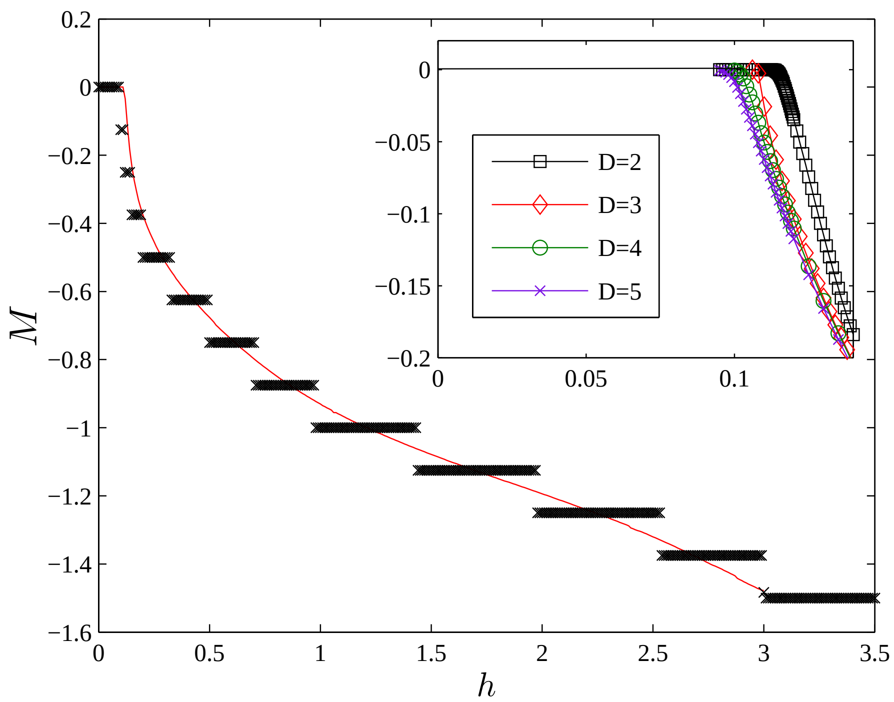
$$H_{\mathrm{AKLT}} = \sum_{\text{edge } \langle i,j \rangle} \hat{P}_{i,j}^{(S=3)} = \frac{27}{160} \sum_{\text{edge } \langle i,j \rangle} \left[\vec{S}_i \cdot \vec{S}_j + \frac{116}{243} (\vec{S}_i \cdot \vec{S}_j)^2 + \frac{16}{243} (\vec{S}_i \cdot \vec{S}_j)^3 + \frac{55}{108} \right]$$

Energy per spin



➔ Gap $\Delta \approx 0.10$

Magnetic moment per spin



Summary and outlook

- Several AKLT states on 2D lattices provide resources for universal quantum computation
- AKLT Hamiltonians on the honeycomb (and square) are gapped (numerical evidence)
- Spin-2 AKLT state on square lattice universal?

- References:

  Garcia-Saez, Murg & **Wei**, in preparation

  Wei, arXiv:1306.1420

  Wei, Affleck & Raussendorf, Phys. Rev. A **86**, 032328 (2012)

  Raussendorf & **Wei**, Annual Review of Cond. Mat. Phys., vol.**3**, 239 (2012)

  Wei, Affleck & Raussendorf, Phys. Rev. Lett. **106**, 070501 (2011)